Set theory - Winter semester 2016-17

Problems	Prof. Peter Koepke
Series 13	Dr. Philipp Schlicht

Problem 49 (2 points)**.** Show that there is no normal non-principal ultrafilter on ω.

Problem 50 (2 points)**.** Show that the club filter

$$\mathcal{C}_{\omega_1} = \{X \subseteq \omega_1 \mid \text{ there is a club } C \subseteq X \text{ in } \omega_1\}$$

on ω_1 is not an ultrafilter.

Problem 51 (6 points)**.** Suppose that κ is a cardinal. If U is a subset of $P(\kappa)$ and $f\colon \kappa \to \kappa$, let

$$f[\![U]\!] = \{X \subseteq \kappa \mid f^{-1}[X] \in U\}.$$

Prove the following statements.

(1) If U is a filter on κ and $f\colon \kappa \to \kappa$, then $f[\![U]\!]$ is a filter on κ.
(2) If U is an ultrafilter on κ and $f\colon \kappa \to \kappa$, then $f[\![U]\!]$ is an ultrafilter on κ.
(3) If U is a normal ultrafilter on κ, then there is a bijection $f\colon \kappa \to \kappa$ such that $f[\![U]\!]$ is not normal.

Problem 52 (4 points)**.** Suppose that $\kappa > \omega_1$ is a regular cardinal. If S is a stationary subset of κ, the *trace* of S is defined as

$$\mathrm{Tr}(S) = \{\alpha < \kappa \mid \mathrm{cof}(\alpha) > \omega \text{ and } S \cap \alpha \text{ is stationary}\}.$$

Prove the following statements for all stationary subsets S and T of κ.

(1) If $S \subseteq T$, then $\mathrm{Tr}(S) \subseteq \mathrm{Tr}(T)$.
(2) $\mathrm{Tr}(S \cup T) = \mathrm{Tr}(S) \cup \mathrm{Tr}(T)$.
(3) $\mathrm{Tr}(\mathrm{Tr}(S)) \subseteq \mathrm{Tr}(S)$.
(4) If $S \triangle T$ is a non-stationary subset of κ, then $\mathrm{Tr}(S) \triangle \mathrm{Tr}(T)$ is a non-stationary subset of κ.

Problem 53 (6 points)**.** Suppose that U is a non-principal ultrafilter on κ. We define the *ultrapower* $\mathrm{Ult}(V,U)$ as the class of all functions $f\colon\kappa\to V$ and the relations $=_U$, $\in_U$ on $\mathrm{Ult}(V,U)$ by

$$f =_U g \iff \{\alpha<\kappa \mid f(\alpha)=g(\alpha)\}\in U,$$

$$f \in_U g \iff \{\alpha<\kappa \mid f(\alpha)\in g(\alpha)\}\in U.$$

Recall that the relation $\in_U$ is called *well-founded* if there is no sequence $\langle f_n \mid n\in\omega\rangle$ with $f_{n+1}\in_U f_n$ for all $n\in\omega$. Prove the following statements.

(1) $=_U$ is an equivalence relation.
(2) If U is ω_1-complete, then $\in_U$ is well-founded.
(3) If $\in_U$ is well-founded, then U is ω_1-complete *(Hint: show that there is a partition $\langle X_n \mid n\in\omega\rangle$ of κ with $X_n\notin U$ for all $n\in\omega$. For each $k\in\omega$, let f_k be a function on κ such that $f_k(\alpha)=n-k$ for all $\alpha\in X_n$).*

Due Friday, February 03, before the lecture.